GROUPE A

$\Leftarrow$) $\exists x_0 \neq 0$ tq $E = H \oplus \text{Vect}(x_0)$

Par G. $\dim(E) = \dim(H) + \underbrace{\dim(\underbrace{\text{Vect}(x_0)}_{\text{droite}})}_{1}$

donc $\dim(H) = \dim(E) - 1$

$\Rightarrow$) Si H est un hyperplan,

H est un sev de E de dim $n - 1 < \infty$

$\exists G$ sev de E, $E = H \oplus G$

avec $\dim(G) = \dim(E) - \dim(H) = 1$

$$M = \frac{1}{3}\begin{array}{c} \begin{matrix} f(e_1) & f(e_2) \end{matrix} \\ \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \end{array} = \text{Mat}_{\mathcal{B}}(f), \quad \mathcal{B} = (e_1, e_2)$$

$$\begin{cases} f(e_1) = \frac{1}{3}(e_1 - 2e_2) \\ f(e_2) = \frac{1}{3}(-e_1 + 2e_2) \end{cases}$$

$\forall x \in E, \exists ! \lambda, \mu \in \mathbb{R}, x = \lambda e_1 + \mu e_2$

$f(x) = f(\lambda e_1 + \mu e_2)$

$\quad = \lambda f(e_1) + \mu f(e_2)$

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f projecteur ssi $\left| \begin{array}{l} f \in \mathcal{L}(E) \\ f \circ f = f \end{array} \right.$

M matrice de proj ssi $M^2 = M$

$M = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \underset{L_2 \leftarrow L_2 + 2L_1}{\sim} \frac{1}{3}\begin{pmatrix} \boxed{1} & -1 \\ 0 & 0 \end{pmatrix}$

$M^2 = \frac{1}{9}\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}^2 = \frac{1}{9}\begin{pmatrix} 3 & -3 \\ -6 & 6 \end{pmatrix} = M$

$rg(f) = rg(M) = 1$ (thé du rg $\dim(\ker(f)) = 1$)

$E = Im(f) \oplus \ker(f)$

$$f(x)=0 \iff M\times\underset{\substack{\uparrow\\ \text{coordonné}}}{\begin{pmatrix} \\ \end{pmatrix}} = \begin{pmatrix}0\\ \vdots\\ 0\end{pmatrix}$$

$$MX=0$$

Soit $X = x\cdot e_1 + y\cdot e_2 \in E$, $x,y\in\mathbb{R}$

$$f(X)=0 \iff M\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\iff \begin{cases} 1 \\ 3 \\ \end{cases}$$

$$\mathrm{Im}(f)=\mathrm{Vect}\left(f(e_1),\dots,f(e_n)\right)$$

$$\mathrm{Mat}_{\mathcal{B}}(f) = \begin{pmatrix} \begin{pmatrix}\cdot\end{pmatrix} & \begin{pmatrix}\cdot\end{pmatrix} & \begin{pmatrix}\cdot\end{pmatrix} & \begin{pmatrix}\cdot\end{pmatrix} \end{pmatrix}^{\mathcal{B}}$$

$$\mathrm{Mat}_{\mathcal{B}}(f) = \begin{array}{c|ccc|c} & f(e_1) & \cdots & f(e_n) & \\ \hline & \boxed{\begin{matrix} a_{11} \\ \vdots \\ a_{n1} \end{matrix}} & & \vdots & \begin{matrix} e_1 \\ \vdots \\ e_n \end{matrix} \end{array}$$

$\mathcal{B} = (e_1, \dots, e_n)$

$$\mathrm{Im}(f) = \mathrm{Vect}(f(e_1), \dots, f(e_n))$$

$$= \mathrm{Vect}\left(a_{11}e_1 + \dots + a_{1n}e_n \right)$$

$f(X) = 0$ $\qquad M = \mathrm{Mat}_{\mathcal{B}}(f)$

$\Leftrightarrow M \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$ $\qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ valeu con[illegible] de X

$\vdots$ $\qquad X = x_1 \cdot e_1 + \dots + x_n e_n$

$$\Leftrightarrow \begin{cases} x_1 = x_1 \\ x_2 = -3x_1 \end{cases} \qquad \mathcal{Y} = \mathrm{Vect}((1, -3))$$

$$X = x_1(1 \cdot e_1 - 3e_2)$$

$M = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix}$

$rg(f) = 1$

$\dim(\ker(f)) = 1$

Soit $X = x\, e_1 + y \cdot e_2 \in E$

$f(X) = 0 \iff M\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \begin{cases} x - y = 0 \\ -2x + 2y = 0 \end{cases}$

$\iff \begin{cases} x = y \\ y = y \end{cases} \iff X = y(e_1 + e_2),\ y \in \mathbb{R}$

$\iff X \in \ker(f)$

$\boxed{\ker f = \mathrm{Vect}(e_1 + e_2)}$

On pourrait aussi $M\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ donc $\mathrm{Vect}(e_1 + e_2) \underset{=}{\subset} \ker f$

$\boxed{\mathrm{Im}(f) = \mathrm{Vect}(e_1 - 2e_2}\ )$

$B_2 = \begin{vmatrix} a+b & a \\ b & a+b \end{vmatrix} = (a+b)^2 - ab$

$= a^2 + ab + b^2$

$B_2 = \frac{a^3 - b^3}{a - b}$

$$f : \begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ (x,y,z) \mapsto (x-y\ ,\ -x+z,\ 3x-2y-z\ ) \end{cases}$$

Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, Y = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3\ ,\ \lambda \in \mathbb{R},$

$$f(\lambda X + Y) = f\left(\begin{pmatrix} \lambda x + a \\ \lambda y + b \\ \lambda z + c \end{pmatrix}\right) = \begin{pmatrix} \lambda x + a - (\lambda y + b) \\ -(\lambda x + a) + (\lambda z + c) \\ 3(\lambda x + a) - 2(\lambda y + b) - (\lambda z + \end{pmatrix}$$

$$= \lambda \cdot \begin{pmatrix} x - y \\ -x + z \\ 3x - 2y - z \end{pmatrix} + \begin{pmatrix} a - b \\ -a + c \\ \dots \end{pmatrix}$$

$$= \lambda f(X) + f(Y)$$

$f \in \mathcal{L}(\mathbb{R}^3)$

$$\mathcal{B}_{can}(\mathbb{R}^3) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + y \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$f\begin{pmatrix}1\\0\end{pmatrix}$

$$\mathrm{Mat}_{\mathcal{B}_{can}}(f) = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 3 & -2 & -1 \end{pmatrix} = M$$

$f((1,0,0)) = (1,-1,3)$

$f \circ g \qquad f^2 = f \circ f$

$$f^n = \underbrace{f \circ f \circ \cdots \circ f}_{n \text{ fois}}$$

$\underline{\det(f \circ f) = \det(f) \times \det(f)}$

$$\mathrm{Im}(f) = \mathrm{Vect}\left(\begin{pmatrix}1\\-1\\3\end{pmatrix}, \begin{pmatrix}-1\\0\\-2\end{pmatrix}, \begin{pmatrix}0\\1\\-1\end{pmatrix} \right)$$

$$= \mathrm{Vect}\left(\begin{pmatrix}1\\-1\\3\end{pmatrix}, \begin{pmatrix}-1\\0\\-2\end{pmatrix} \right)$$

$$M = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 0 & 1 \\ 3 & -2 & -1 \end{pmatrix}$$

Soit $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \ker f \iff M \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \iff \begin{cases} x - y = 0 \\ -x + z = 0 \\ 3x - 2y + z = 0 \end{cases}$$

$$\iff \begin{cases} y = x \\ z = x \\ x = x \end{cases} \iff \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$\ker f = \mathrm{Vect}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right)$ donc $\mathrm{rg}(f) = 2$

4/ $\mathrm{Mat}(f^2) = M^2 = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{pmatrix}$

$\mathrm{Im}(f^2) = \mathrm{Vect}\left(\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}\right) = \ker(f)$

Soit $X \in \mathbb{R}^3$ $\quad f^3(X) = f(\underbrace{f^2(X)}) = 0$ $\quad (\in \mathrm{Im}(f^2) = \ker(f))$

$\forall n \geq 3$, f^n est l'appli nulle.

$$f(A) \subset f(B) \iff A + \ker(f) \subset B + \ker(f)$$

$\Rightarrow$ | On suppose $\underline{f(A) \subset f(B)} \iff \forall a \in A \; f(a) \in f(B)$

$\iff \forall a \in A, \exists b \in B \; f(a) = f(b)$

Soit $x \in A + \ker(f)$

$\exists a \in A, x_k \in \ker(f) \quad x = a + x_k$

$f(x) = f(a) + f(x_k) = f(a)$

$\exists b \in B \quad f(x) = f(b)$

donc $f(\underbrace{x - b}_{\in \ker(f)}) = 0$

:

Donc $x \in B + \ker(f)$

(⇐) On suppose $A + \ker f \subset B + \ker f$

Soit $y \in f(A)$.

$\exists\, \underline{a} \in A, \quad y = f(a) + \underset{\substack{\uparrow \\ \in \ker(f)}}{0}$

⋮

$y \in f(B)$

GROUPE B

H hyperplan $\underline{\text{ssi}}$ $\exists x_0 \neq 0,\ E = H \oplus \mathrm{Vect}(x_0)$

$\Rightarrow$) Soit H un hyperplan de E

H est dim $n-1 < \infty$ où $n = \dim E$

Donc $\exists G$ sev de E, $E = H \oplus G$

Par G., $\dim(G) = \dim(E) - \dim(H)$

$$= n - (n-1) = 1$$

$\exists x_0 \neq 0,\ G = \mathrm{Vect}(x_0)$

$\forall x_0 \in G,\ x_0 \neq 0$

$\Leftarrow$) On suppose $\exists x_0 \neq 0,\ E = H \oplus \mathrm{Vect}(x_0)$

Par G. $\dim(E) = \dim(H) + \underbrace{\dim(\mathrm{Vect}(x_0))}_{1}$

$$M = \mathrm{Mat}_{\mathcal{B}}(f) = \frac{1}{3} \begin{array}{c} \begin{array}{cc} f(e_1) & f(e_2) \end{array} \\ \begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ e_2 \end{array}$$

$\mathcal{B} = (e_1, e_2)$

$$f(e_1) = \frac{1}{3}(e_1 - 2e_2)$$

$$f(e_2) = \frac{1}{3}(-e + 2e_2)$$

Soit $x \in E$, $x = \lambda \cdot e_1 + \mu e_2$ $\exists ! \lambda, \mu \in \mathbb{R}$

$$f(x) = \lambda f(e_1) + \mu f(e_2)$$

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f est proj ssi $\begin{cases} f \in \mathcal{L}(E) \\ f \circ f = f \end{cases}$

M est la matrice d'une proj ssi $M^2 = M$

$$\text{Mat}_B(f \circ g) = \text{Mat}(f) \times \text{Mat}(g)$$

$$rg(f) = rg(M)$$

$$M = \frac{1}{9}\begin{pmatrix}1 & -1\\ -2 & 2\end{pmatrix}^2 = \frac{1}{9}\begin{pmatrix}3 & -3\\ -6 & 6\end{pmatrix} = \frac{1}{3}\begin{pmatrix}1 & -1\\ -2 & 2\end{pmatrix} = M$$

M est bien la matrice d'une proj.

$E = \mathrm{Im}(f) \oplus \mathrm{Ker}(f)$

$\mathrm{rg}(M) = \mathrm{rg}(f) = 1$ $\quad$ $\dim(\mathrm{Ker}(f)) = 1$

~~$\mathrm{rg}(f(e_1), \dots, f(e_n))$~~

$\mathrm{Im}(f) = \mathrm{Vect}(f(e_1), \dots, f(e_n))$

$$\mathrm{Mat}_{\mathcal{B}}(f) = \begin{array}{c} \begin{array}{ccc} f(e_1) & f(e_2) & f(e_n) \end{array} \\ \begin{pmatrix} \bullet & & \\ \bullet & & \\ \vdots & & \\ \bullet & & \end{pmatrix} \end{array} \begin{array}{c} e_1 \\ \vdots \\ e_n \end{array}$$

Soit $x \in E$, $x = x_1 \cdot e_1 + \dots + x_n \cdot e_n$

$$X = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \leftarrow \text{vecteur coordonné}$$

$f(x) = 0 \iff MX = 0_n$

$MX = 0$

$$M = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -2 & 2 \end{pmatrix} \qquad \mathcal{B} = (e_1, e_2)$$

Rq $e_1 + e_2 \in k(f)$

$\mathrm{Vect}(e_1 + e_2) \subset \ker(f)$

$f(x) = 0$

Soit $x \in E$, $x = x_1 \cdot e_1 + x_2 \cdot e_2$

$$\mathrm{Mat}_{\mathcal{B}}(x) = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = X \in \mathbb{R}^2$$

$$\mathrm{Mat}(f(x)) = \mathrm{Mat}_{\mathcal{B}}(f) \times \mathrm{Mat}(x)$$

$$= M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$x \in \ker f \iff M \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\iff \begin{cases} x_1 - x_2 = 0 \\ -2x_1 + 2x_2 = 0 \end{cases} \iff (S) \begin{cases} x_1 = x_2 \\ x_2 = x_2 \end{cases}$$

$$S : \mathrm{Vect}\left(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$$

$$\iff \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathrm{Vect}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\iff x = x_2 (e_1 + e_2)$$

$\iff x \in \mathrm{Vect}(e_1 + e_2)$

Ker f sev de E

$$\mathrm{Im}(f) = \mathrm{Vect}(f(e_1), f(e_2))$$
$$= \mathrm{Vect}(e_1 - 2e_2, -e_1 + 2e_2)$$
$$= \mathrm{Vect}(e_1 - 2e_2)$$

$$f : \begin{cases} \mathbb{R}^3 \longrightarrow \mathbb{R}^3 \\ (x, y, z) \longmapsto (x - y, -x + z, 3x - 2y - z) \end{cases}$$

Soit $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, $Y = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{R}^3$, $\lambda \in \mathbb{R}$

$$f(\lambda X + Y) = f\left(\begin{pmatrix} \lambda x + a \\ \lambda y + b \\ \lambda z + c \end{pmatrix}\right)$$

$$= \begin{pmatrix} \lambda x + \underline{a} - (\lambda y + \underline{b}) \\ -(\lambda x + a) + (\lambda z + c) \\ 3(\lambda x + a) - 2(\lambda y + b) - (\lambda z + c) \end{pmatrix}$$

$$= \lambda \cdot \begin{pmatrix} x-y \\ -x+z \\ 3x-2y-z \end{pmatrix} + \begin{pmatrix} a-b \\ -a+c \\ 3a-2b-c \end{pmatrix}$$

$$= \lambda f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) + f\left(\begin{pmatrix} a \\ b \\ c \end{pmatrix}\right)$$

$$= \lambda f(X) + f(Y)$$

Donc $f \in \mathcal{L}(\mathbb{R}^3)$

$$\mathcal{B}_{can}(\mathbb{R}^3) = \left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$$

$$\mathrm{Mat}_{\mathcal{B}_{can}}(f) = \begin{pmatrix} \overset{f\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}\right)}{1} & -1 & 0 \\ -1 & 0 & 1 \\ 3 & -2 & -1 \end{pmatrix} := M$$

$$f((1,0,0)) = \begin{pmatrix}1\\-1\\3\end{pmatrix} = 1\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} + (-1)\cdot\begin{pmatrix}0\\1\\0\end{pmatrix} + 3\cdot\begin{pmatrix}0\\0\\1\end{pmatrix}$$

Soit $\begin{pmatrix}x\\y\\z\end{pmatrix} = X \in \mathbb{R}^3$

$$f((x,y,z)) = 0 \iff MX = \begin{pmatrix}0\\0\\0\end{pmatrix}$$

$$\Updownarrow$$

$$X \in \ker(f) \qquad \iff \begin{cases} x - y = 0 \\ -x + z = 0 \\ 3x - 2y - z = 0 \end{cases}$$

~~$L_2 \leftarrow L_2 + L_1$~~

~~$L_3 \leftarrow L_3 - 3L_1$~~

$$\iff \begin{cases} y = x \\ z = x \\ \cancel{x} = \cancel{x} \end{cases} \iff \begin{pmatrix}x\\y\\z\end{pmatrix} = x\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$\iff X \in \mathrm{Vect}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)$$

$$\boxed{\ker(f) = \mathrm{Vect}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right)}$$

$\mathrm{rg}(g) = 2$ (thé du rang)

$$\mathrm{Im}(f) = \mathrm{Vect}\left(\underset{\uparrow}{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}}, \underset{\uparrow}{\begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}}, \underset{\uparrow}{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}} \right)$$

$$= \mathrm{Vect}\left(\underbrace{\begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \end{pmatrix}}_{\text{base de } \mathrm{Im}(f)} \right)$$

$$\begin{cases} y = 2x + t \\ z = -x + t \\ \boxed{\begin{matrix} x = x \\ t = t \end{matrix}} \end{cases} \qquad \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} \in \mathrm{Vect}\left(\begin{pmatrix} 1 \\ 2 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix} \right)$$

$$\boxed{\text{'' base can on ls par ''} \quad \boxed{\begin{pmatrix} x \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} \text{ ou } \boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}}$$

[illegible]

$\boxed{0 = 0}$

$$\begin{cases} \boxed{x} + 2y + 3z + 5t = 0 \\ \boxed{2}y + 3z = 0 \\ \boxed{t} = 0 \\ z = z \end{cases}$$

$$\begin{pmatrix} \boxed{1} & 2 & 3 & 5 \\ 0 & \boxed{2} & 3 & 0 \\ 0 & 0 & 0 & \boxed{1} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$f^2 = f \circ f$

$f^n = \underbrace{f \circ f \circ \dots \circ f}_{n \text{ fois}}$

$\mathrm{Mat}(f^2) = \mathrm{Mat}(f) \times \mathrm{Mat}(f)$

$= M^2 = \begin{pmatrix} 2 & -1 & -1 \\ 2 & -1 & -1 \\ 2 & -1 & -1 \end{pmatrix}$

$$\mathrm{Im}(f^2) = \mathrm{Vect}\left(\begin{pmatrix}1\\1\\1\end{pmatrix}\right) = \ker f$$

$$\forall x \in E \;,\; f^3(x) = f(\underbrace{f^2(x)}_{\in \mathrm{Im}(f^2) = \ker(f)}) = 0$$

f^3 est l'appli nulle

$$\forall n \geqslant 3 \qquad f^n = 0_{\mathcal{L}(\mathbb{R}^3)}$$